CAREER POINT

JEE Advanced Exam 2016
(Paper & Solution)

Date : 22 / 05 / 2016

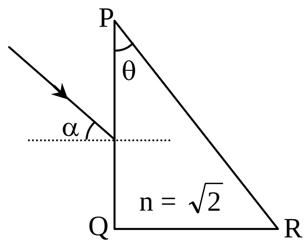
PART I - PHYSICS

SECTION – 1 (Maximum Marks : 15)

- This section contains **FIVE** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct.
- For each question, darken the bubble corresponding to the correct option in the ORS.
- For each question, marks will be awarded in one of the following categories:

Full Marks	: +3	If only the bubble corresponding to the correct option is darkened.
Zero Marks	: 0	If none of the bubbles is darkened.
Negative Marks	: –1	In all other cases.

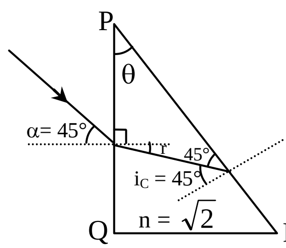
Q.1 A parallel beam of light is incident form air at an angle α on the side PQ of a right angled triangular prism of refractive index n = $\sqrt{2}$. Light undergoes total internal reflection in the prism at the face PR when α has a minimum value of 45°. The angle θ of the prism is



(A) 15° (B) 22.5° (C) 30° (D) 45°

Ans. **[A]**

Sol.



$\sin i_C = \frac{1}{\sqrt{2}}$

$i_C = 45°$

$1 \times \sin 45° = \sqrt{2}\ \sin r$

$\sin r = \frac{1}{2}$

$r = 30°$

$\therefore\ \theta = 180° – 120° – 45° = 15°$

Q.2 In a historical experiment to determine Plank's constant, a metal surface was irradiated with light of different wavelengths. The emitted photoelectron energies were measured by applying a stopping potential. The relevant data for the wavelength (λ) of incident light and the corresponding stopping potential (V_0) are given below.

λ (μm)	V_0 (Volt)
0.3	2.0
0.4	1.0
0.5	0.4

Given that $c = 3 \times 10^8$ m s^{-1} and $e = 1.6 \times 10^{-19}$ C, Planck's constant (in units of J s) found from such an experiment is

(A) 6.0×10^{-34} (B) 6.4×10^{-34} (C) 6.6×10^{-34} (D) 6.8×10^{-34}

Ans. **[B]**

Sol. We have, $(KE)_{max} = \frac{hc}{\lambda} - \phi = eV_0$

$\frac{hc}{\lambda} = \phi + eV_0$

$\frac{hc}{3\times10^{-7}} = \phi + eV_0$

$\frac{h\times3\times10^{8}}{3\times10^{-7}} = \phi + 1.6 \times 10^{-19} \times 2$

$h \times 10^{15} = \phi + 3.2 \times 10^{-19}$ ......(i)

$\frac{h\times3\times10^{8}}{4\times10^{-7}} = \phi + 1.6 \times 10^{-19} \times 1$

$0.75\ h \times 10^{15} = \phi + 1.6 \times 10^{-19}$ ......(ii)

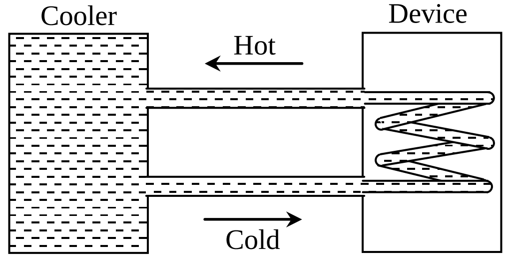
by equation (i) – (ii)

$0.25\ h \times 10^{15} = 1.6 \times 10^{-19}$

$h = \frac{1.6\times10^{-34}}{0.25}$

$h = 6.4 \times 10^{-34}$

Q.3 A water cooler of storage capacity 120 liters can cool water at a constant rate of P watts. In a closed circular system (as shown schematically in the figure), the water from the cooler is used to cool an external device that generates constantly 3 kW of heat (thermal load). The temperature of water fed into the device cannot exceed 30° C and the entire stored 120 litres of water is initially cooled to 10° C. The entire system is thermally insulated. The minimum value of P (in watts) for which the device can be operated for 3 hours is



(Specific heat of water is 4.2 kJ kg^{-1} K^{-1} and the density of water is 1000 kg m^{-3})

(A) 1600 (B) 2067 (C) 2533 (D) 3933

Ans. **[B]**

Sol. Rate of heat extracted by water = $\left(\frac{ms\Delta T}{\Delta t}\right)$

Power by cooler + $\frac{ms\Delta T}{\Delta t}$ = H

P + $\frac{ms\Delta T}{\Delta t}$ = H

For P_{min} in Δt = 3 hrs.

$P = H - \frac{ms\Delta T}{\Delta t}$

$P = 3 \times 10^3 - \frac{\rho \times 120 \times 10^{-3} \times 4.2 \times 10^3 \times (30-10)}{3 \times 60 \times 60}$

P = 2067 Watt

Q.4 A uniform wooden stick of mass 1.6 kg and length ℓ rests in an inclined manner on a smooth, vertical wall of height h (<ℓ) such that a small portion of the stick extends beyond the wall. The reaction force of the wall on the stick is perpendicular to the stick. The stick makes an angle 30° with the wall and the bottom of the stick is on a rough floor. The reaction of the wall on the stick is equal in magnitude to the reaction of the floor on the stick. The ratio $\frac{h}{\ell}$ and the frictional force f at the bottom of the stick are

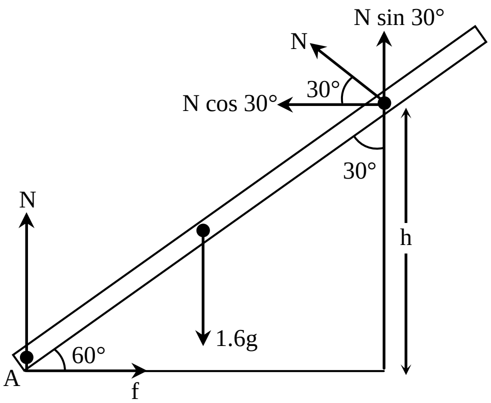
(g = 10 m s^{-2})

(A) $\frac{h}{\ell} = \frac{\sqrt{3}}{16}$, $f = \frac{16\sqrt{3}}{3}$ N (B) $\frac{h}{\ell} = \frac{3}{16}$, $f = \frac{16\sqrt{3}}{3}$ N

(C) $\frac{h}{\ell} = \frac{3\sqrt{3}}{16}$, $f = \frac{8\sqrt{3}}{3}$ N (D) $\frac{h}{\ell} = \frac{3\sqrt{3}}{16}$, $f = \frac{16\sqrt{3}}{3}$ N

Ans. [D]

Sol.



$N \sin 30° + N = 1.6\ g$

$\frac{3N}{2} = 1.6\ g \Rightarrow N = \frac{3.2}{3} g$ ....(i)

$f = N \cos 30° = \frac{3.2}{3} g \times \frac{\sqrt{3}}{2} = \frac{16\sqrt{3}}{2}\ N$ ......(ii)

$\tau_A = 1.6 \times \frac{\ell}{2} \sin(90 + 60) - N(x) \sin 90° = 0$

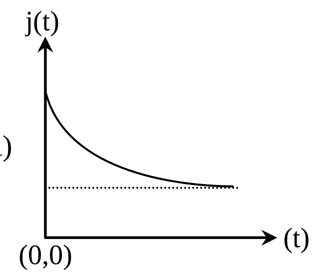
$\Rightarrow 1.6\ g \times \frac{\ell}{2} \times \frac{1}{2} = \quad Nx = \frac{32}{3} x$

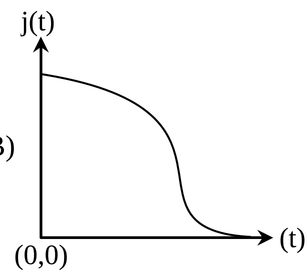
$4\ell = \frac{32}{3} x$ $\quad [\frac{h}{x} = \sin 60°]$

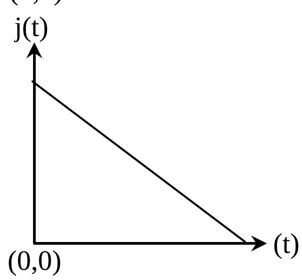
$4\ell = \frac{32}{3} \times \frac{h}{\sqrt{3}} \times 2$

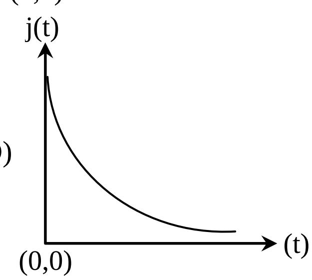
$\frac{h}{\ell} = \frac{3\sqrt{3}}{16}$

Q.5 An infinite line charge of uniform electric charge density λ lies along the axis of an electrically conducting infinite cylindrical shell of radius R. At time t = 0, the space inside the cylinder is filled with a material of permittivity ε and electrical conductivity σ. The electrical conduction in the material follows Ohm's law. Which on the following graphs best describes the subsequent variation of the magnitude of current density j(t) at any point in the material .

(A)


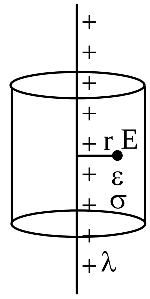
(B)


(C)


(D)


Ans. **[D]**

Sol.



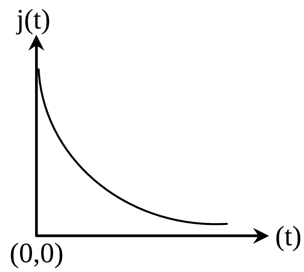
Electric field intensity at a distance r (< R)

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Current density $J = \sigma E$

$$J = \frac{\sigma\lambda}{2\pi\varepsilon_0 r}$$

Due to discharging, charge density (λ) decreases exponentially with time till it become zero.
therefore answer is **(D)**



SECTION – 2 (Maximum Marks : 32)

- This section contains **EIGHT** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four options is (are) correct.
- For each question, darken the bubble (s) corresponding to all the correct option (s) in the ORS.
- For each question, marks will be awarded in one of the following categories:

Full Marks	: +4	If only the bubble (s) corresponding to all the correct option (s) is (are) darkened.
Partial Marks	: +1	For darkening a bubble corresponding **to each correct option**, provided NO incorrect option is darkened.
Zero Marks	: 0	If none of the bubbles is darkened.
Negative Marks	: –2	In all other cases.

- For example, If (A), (C) and (D) are all the correct options for a question, darkening all these three will result in +4 marks ; darkening only (A) and (D) will result in +2 marks ; and darkening (A) and (B) will result in –2 marks, as a wrong option is also darkened

Q.6 A plano-convex lens is made of a material of refractive index n. When a small object is placed 30 cm away in front of the curved surface of the lens, and image of the double the size of the object is produced. Due to reflection from the convex surface of the lens, another faint image is observed at a distance of 10 cm away from the lens. Which of the following statement (s) is (are) true ?

(A) The refractive index of the lens is 2.5

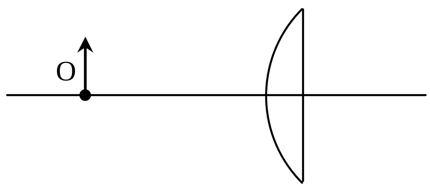
(B) The radius of curvature of the convex surface is 45 cm

(C) The faint image is erect and real

(D) The focal length of the lens is 20 cm

Ans. **[A,D]**

Sol.



for Refraction through lens

$u = -30$ cm

$m = -2$

$$m = \frac{f}{f+u}$$

$$-2 = \frac{f}{f-30}$$

$-2f + 60 = f$

$f = 20$ cm

for Reflection through convex surface

$u = -30$, $v = 10$

$$\frac{1}{10} + \frac{1}{-30} = \frac{1}{f_0}$$

$f_0 = 15$

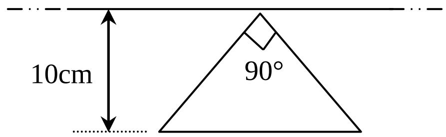
$R = 2f_0 = 30$

Now for plano-convex lens

$$\frac{1}{f} = \frac{(n-1)}{R}$$

$$\frac{1}{20} = \frac{(n-1)}{30} \Rightarrow n = 2.5$$

Q.7 A conducting loop in the shape of a right angled isosceles triangle of height 10 cm is kept such that the 90° vertex is very close to an infinity long conducting wire (see the figure). The wire is electrically insulated from the loop. The hypotenuse of the triangle is parallel to the wire. The current in the triangular loop is in counterclockwise direction and increased at a constant rate of 10 As^{-1}. Which of the following statement (s) is (are) true ?



(A) There is a repulsive force between the wire and the loop

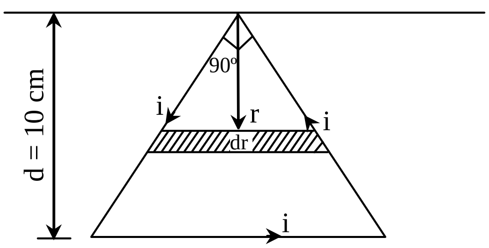
(B) If the loop is rotated at a constant angular speed about the wire, an additional emf of $\left(\frac{\mu_0}{\pi}\right)$ volt is induced in the wire

(C) The magnitude of induced emf in the wire is $\left(\frac{\mu_0}{\pi}\right)$ volt

(D) The induced current in the wire is in opposite direction to the current along the hypotenuse

Ans. **[A,C]**

Sol.



Here $\frac{di}{dt} = 10$ A/S

for mutual inductance

$d\phi = \frac{\mu_0}{2\pi}\frac{i}{r}(2r\,dr)$

$\phi = \frac{\mu_0}{\pi} i\int_0^d dr$

$\phi = \left(\frac{\mu_0}{\pi}d\right)i = Mi$

$M = \frac{\mu_0}{\pi}d = 0.1\frac{\mu_0}{\pi}$

When current through loop increases then magnetic flux

$\phi = Mi$

$\varepsilon = \frac{d\phi}{dt}$

$\varepsilon = M\frac{di}{dt} = \frac{\mu_0}{\pi} \times 10 \times 0.1$

$\varepsilon = \frac{\mu_0}{\pi}$

Force between loop and conductor is repulsive.

Q.8 The position vector $\vec{r}$ of a particle of mass m is given by the following equation

$$\vec{r}(t) = \alpha t^3\,\hat{i} + \beta t^2\,\hat{j},$$

where $\alpha = \frac{10}{3}$ ms^{-3}, $\beta = 5$ ms^{-2} and m = 0.1 kg. At t = 1 s, which of the following statement (s) is (are) true about the particle ?

(A) The velocity $\vec{v}$ is given by $\vec{v} = (10\hat{i} + 10\hat{j})$ ms^{-1}

(B) The angular momentum $\vec{L}$ with respect to the origin is given by $\vec{L} = -\left(\frac{5}{3}\right)\hat{k}$ N ms

(C) The force $\vec{F}$ is given by $\vec{F} = (\hat{i} + 2\hat{j})$ N

(D) The torque $\vec{\tau}$ with respect to the origin is given by $\vec{\tau} = -\left(\frac{20}{3}\right)\hat{k}$ Nm

Ans. **[A,B,D]**

Sol. $\vec{r}(t) = \alpha t^3\,\hat{i} + \beta t^2\,\hat{j}$

$\alpha = \frac{10}{3}$ ms^{-3}, $\beta = 5$ ms^{-2} and m = 0.1 kg. t = 1 s,

$\vec{v} = 3\alpha t^2\,\hat{i} + 2\beta t\,\hat{j}$

(A) $(\vec{v})_{t=1} = 3\alpha\hat{i} + 2\beta\,\hat{j}$

$= 3 \times \frac{10}{3}\hat{i} + 2 \times 5\hat{j}$

$= 10\hat{i} + 10\hat{j}$

(B) $\vec{L} = \vec{r} \times \vec{P} = m(\vec{r} \times \vec{v})$

$= 0.1\,[(\alpha t^3\,\hat{i} + \beta t^2\hat{j}) \times (10\hat{i} + 10\hat{j})]$

$= 0.1\,[(10\alpha t^3\,(\hat{k}) - 10\,\beta t^2\,(\hat{k})]$

$= 0.1\,[\,10 \times \frac{10}{3} \times 1\hat{k} - 10 \times 5\,(1)^2\,\hat{k}\,]$

$= 0.1\,[\,\frac{100}{3} - 50\,]\,\hat{k}$

$= -\frac{5}{3}\,(\hat{k})$

(C) $\vec{F} = m\,\vec{a} = m\,[6\,\alpha t\,\hat{i} + 2\beta\,\hat{j}]$

$0.1\,[6 \times \frac{10}{3} \times 1\,\hat{i} + 2 \times 5\,\hat{j}]$

$= [2\hat{i} + \hat{j}]$

(D) $\vec{\tau} = \vec{r} \times \vec{F}$

$= (\alpha t^3\,\hat{i} + \beta t^2\,\hat{j}) \times [2\hat{i} + \hat{j}]$

$= (\frac{10}{3}\,\hat{i} + 5\,\hat{j}) \times [2\hat{i} + \hat{j}]$

$= \frac{10}{3}\,\hat{k} - 10\,\hat{k}$

$= \frac{-20}{3}\,\hat{k}$

Q.9 A length-scale (ℓ) depends on the permittivity (ε) of a dielectric material, Boltzmann constant (k_B), the absolute temperature (T), the number per unit volume (n) of certain charged particles, and the charge (q) carried by each of the particles. Which of the following expressions (s) for ℓ is (are) dimensionally correct ?

(A) $\ell = \sqrt{\left(\frac{nq^2}{\varepsilon k_B T}\right)}$

(B) $\ell = \sqrt{\left(\frac{\varepsilon k_B T}{nq^2}\right)}$

(C) $\ell = \sqrt{\left(\frac{q^2}{\varepsilon n^{2/3} k_B T}\right)}$

(D) $\ell = \sqrt{\left(\frac{q^2}{\varepsilon n^{1/3} k_B T}\right)}$

Ans. **[B,D]**

Sol. $\ell \ \alpha \ \varepsilon^a k^b T^c n^d q^e$

(A) $\ell = \sqrt{\frac{L^{-3} \times A^2T^2}{M^{-1}A^2T^4L^{-3}M^1L^2T^{-2}\theta^{-1}\theta}}$

$\ell = \sqrt{\frac{1}{L^2}} = \frac{1}{L}$

(B) $\ell = \sqrt{\frac{\varepsilon\ k_B\ T}{nq^2}}$

$= \sqrt{\frac{(M^{-1}A^2T^4L^{-3})M^1L^2T^{-2}\theta^{-1}\theta}{L^{-3}A^1T^2}}$

$= \sqrt{L^2} = L$

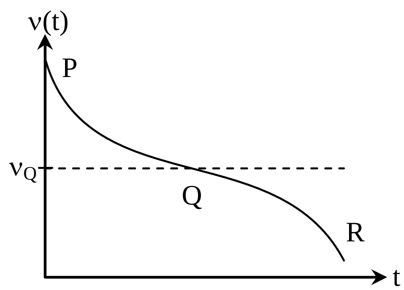
(C) $\ell = \sqrt{\frac{A^2T^2}{M^{-1}A^2T^4L^{-3}L^{-2}M^1L^2T^{-2}\theta^{-1}\theta}}$

(D) $\ell = \sqrt{\frac{A^2T^2}{M^{-1}A^2T^4L^{-3}L^{-1}M^{+1}L^2T^{-2}\theta^{-1}\theta}}$

$= \sqrt{L^2} = L$

Q.10 Two loudspeakers M and N are located 20 m apart and emit sound at frequencies 118 Hz and 121 Hz, respectively. A car is initially at a point P, 1800 m away from the midpoint Q of the line MN and moves towards Q constantly at 60 km/hr along the perpendicular bisector of MN. It crosses Q and eventually reaches a point R, 1800 m away from Q. Let $\nu(t)$ represent the beat frequency measured by a person sitting in the car at time t. Let ν_P, ν_Q and ν_R be the beat frequencies measured at location P, Q and R, respectively. The speed of sound in air is 330 ms^{-1}. Which of the following statement(s) is(are) true regarding the sound heard by the person ?

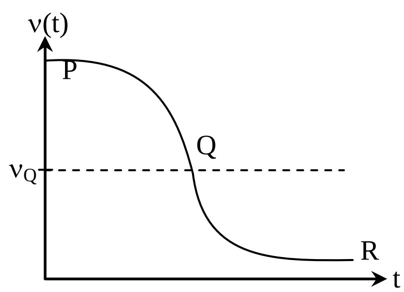
(A) The plot below represents schematically the variation of beat frequency with time



(B) The rate of change in beat frequency is maximum when the car passes through Q

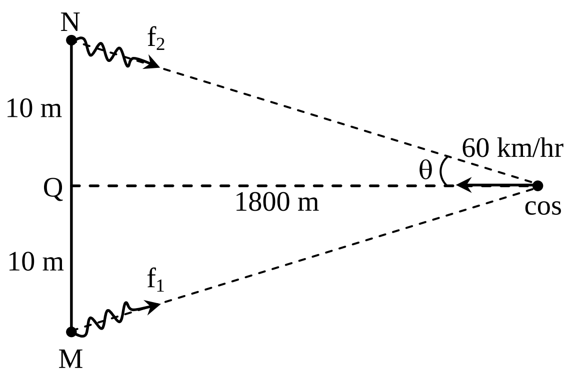
(C) $\nu_P + \nu_R = 2\nu_Q$

(D) The plot below represents schematically the variation of beat frequency with time



Ans. **[B,C,D]**

Sol.



Beat frequencies in $f_B = \frac{V + V_0 \cos\theta}{V}$ $[f_2 - f_1]$

∵ cos θ decreases with time because θ increases.

So f_B decreases.

$$\frac{df_B}{dt} = \frac{d}{dt}\left[1 + \frac{V_0}{V}\cos\theta\right](f_2 - f_1)$$

$$= \left[0 + \frac{V_0}{V}(\sin\theta)\left(\frac{d\theta}{dt}\right)\right](f_2 - f_1)$$

$\because$ θ increases so sin θ also increase from 0 to 90º and slope increases.

So graph is (D).

Rate of change in beat frequencies is maximum where θ = 90º or at Q. (B) is correct.

At P beats frequencies $\nu_P = \frac{V + V_0}{V}[f_2 - f_1]$

At R beats frequencies $\nu_R = \frac{V - V_0}{V}[f_2 - f_1]$

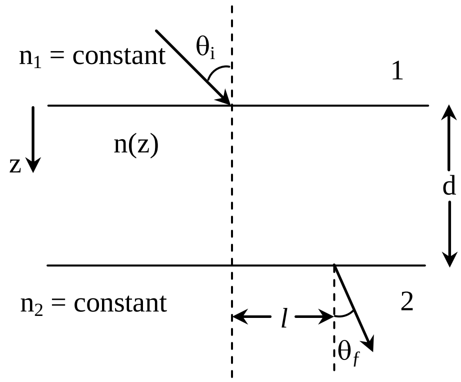
$$\nu_P + \nu_R = \left(\frac{V + V_0}{V} + \frac{V - V_0}{V}\right)[f_2 - f_1]$$

$$= 2(f_2 - f_1)$$

$\nu_P + \nu_R = 2\nu_Q$

(C) Also correct.

Q.11 A transparent slab of thickness d has a refractive index n (z) that increases with z. Here z is the vertical distance inside the slab, measured from the top. The slab is placed between two media with uniform refractive indices n_1 and n_2 (> n_1), as shown in the figure. A ray of light is incident with angle θ_i from medium 1 and emerges in medium 2 with refraction angle θ_f with a lateral displacement *l*.



Which of the following statement(s) is(are) true ?

(A) *l* is dependent on n(z)

(B) $n_1 \sin\theta_i = (n_2 - n_1)\sin\theta_f$

(C) $n_1 \sin\theta_i = n_2 \sin\theta_f$

(D) *l* is independent of n_2

Ans. **[A,C,D]**

Sol. By Snell's Law

$n_1 \sin\theta_i = n_2 \sin\theta_f$

l will depend on angle of refraction which depends on refractive index

l does not depend on n_2 as *l* is defined upto the instant when ray emerge from slab.

Q.12 Highly excited states for hydrogen-like atoms (also called Rydberg states) with nuclear charge Ze are defined by their principal quantum number n, where n >> 1. Which of the following statement(s) is(are) true ?

(A) Relative change in the radii of two consecutive orbitals does not depend on Z

(B) Relative change in the radii of two consecutive orbitals varies as l/n

(C) Relative change in the energy of two consecutive orbitals varies as $1/n^2$

(D) Relative change in the angular momenta of two consecutive orbitals varies as 1/n

Ans. **[A,B,D]**

Sol. $r = 0.53\,\frac{n^2}{z}$ Å

$$r_{n+1} - r_n = \frac{0.53}{z}[(n+1)^2 - n^2]$$

$$\text{relative change} = \frac{0.53}{z}\left[\frac{(n+1)^2 - n^2}{\frac{n^2\,(0.53)}{z}}\right]$$

$$= 0.53\left[\left(1+\frac{1}{n}\right)^2 - 1\right]$$

$$= 0.53\left[1+\frac{2}{n}-1\right] \Rightarrow \frac{0.53\times 2}{n} \qquad \text{option (A,B)}$$

$$\frac{E_{n+1}-E_n}{E_n} = -\frac{13.6z^2}{\frac{13.6z^2}{n^2}}\left[\frac{1}{n^2}-\frac{1}{(n+1)^2}\right]$$

$$= \frac{n^2}{n^2}\left[1-\frac{1}{\left(1+\frac{1}{n}\right)^2}\right]$$

$$= 1 - \left(1+\frac{1}{n}\right)^{-2}$$

$$= 1 - \left(1-\frac{2}{n}\right) = \frac{2}{n}$$

$$\frac{L_{n+1}-L_n}{L_n} \Rightarrow \frac{(n+1)\frac{h}{2\pi} - \frac{nh}{2\pi}}{\frac{nh}{2\pi}} \Rightarrow \frac{1}{n}$$

Q.13 An incandescent bulb has a thin filament of tungsten that is heated to high temperature by passing an electric current. The hot filament emits black-body radiation. The filament is observed to break up at random locations after a sufficiently long time of operation due to non-uniform evaporation of tungsten from the filament. If the bulb is powered at constant voltage, which of the following statement(s) is(are) true ?

(A) The temperature distribution over the filament is uniform

(B) The resistance over small sections of the filament decreases with time

(C) The filament emits more light at higher band of frequencies before it breaks up

(D) The filament consumes less electrical power towards the end of the life of the bulb

Ans. **[D]**

SECTION – 3 (Maximum Marks 15)

- This section contains FIVE questions.
- The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- For each question, marks will be awarded in one of the following categories.

  Full Marks : +3 If only the bubble corresponding to the correct answer is darkened.

  Zero Marks : 0 In all other cases.

Q.14 A hydrogen atom in its ground state is irradiated by light of wavelength 970 Å. Taking $hc/e = 1.237 \times 10^{-6}$ eV m and the ground state energy of hydrogen atom as – 13.6 eV, the number of lines present in the emission spectrum is.

Ans. **[6]**

Sol. Z = 1

n = 1

$\lambda = 970$

$$E = \frac{12400\,eV}{970} = 12.75\ eV$$

Energy difference between n = 1 & n = 4.

So electron transit to n = 4

$$\therefore \text{ no. of lines in emission spectrum } = \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6.$$

Q.15 The isotope ${}^{12}_{5}B$ having a mass 12.014 u undergoes β-decay to ${}^{12}_{6}C$. ${}^{12}_{6}C$ has an excited state of the nucleus (${}^{12}_{6}C^{*}$) at 4.041 MeV above its ground state. If ${}^{12}_{5}B$ decays to ${}^{12}_{6}C^{*}$, the maximum kinetic energy of the β-particle in units of MeV is. (1 u = 931.5 MeV/c^2, where c is the speed of light in vacuum).

Ans. **[9]**

Sol. $^{12}_{5}B \longrightarrow {}^{12}_{6}C + {}^{0}_{-1}\beta$

Mass loss = 0.014 u

Energy produced = 13.041 MeV ($E = mc^2$)

$\therefore$ Maximum Kinetic energy of β-particle = 13.041 – 4.041 = 9 MeV

Q.16 Consider two solid spheres P and Q each of density 8 gm cm^{-3} and diameters 1 cm and 0.5 cm, respectively. Sphere P is dropped into a liquid of density 0.8 gm cm^{-3} and viscosity η = 3 poiseulles. Sphere Q is dropped into a liquid of density 1.6 gm cm^{-3} and viscosity η = 2 poiseulles. The ratio of the terminal velocities of P and Q is.

Ans. **[3]**

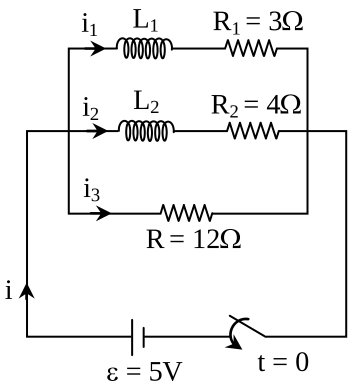
Sol. $v = \frac{2}{9}\,\frac{r^2(d-\rho)g}{\eta}$

$$\frac{v_P}{v_Q} = \frac{\frac{2}{9}\left(\frac{1}{2}\right)^2\frac{(8-0.8)}{3}}{\frac{2}{9}\left(\frac{0.5}{2}\right)^2\frac{(8-1.6)}{2}} = \frac{4\times 7.2\times 2}{3\times(6.4)} = 3.$$

Q.17 Two inductors L_1 (inductance 1 mH, internal resistance 3 Ω) and L_2 (inductance 2 mH, internal resistance 4 Ω), and a resistor R (resistance 12 Ω) are all connected in parallel across a 5 V battery. The circuit is switched on at time t = 0. The ratio of the maximum to the minimum current (I_{max} / I_{min}) drawn from the battery is.

Ans. **[8]**

Sol.



at t

$$i_1 = \frac{\varepsilon}{R_1}\left(1 - e^{-\frac{R_1 t}{L_1}}\right)$$

$$i_1 = \frac{5}{3}(1 - e^{-3t})$$

$$i_2 = \frac{5}{4}(1 - e^{-2t})$$

$$i_3 = \frac{5}{12}$$

$$i = i_1 + i_2 + i_3$$

$$i = \frac{5}{3} - \frac{5}{3}e^{-3t} + \frac{5}{4} - \frac{5}{4}e^{-2t} + \frac{5}{12}$$

$i = \frac{10}{3} - \frac{5}{3}e^{-3t} - \frac{5}{4}e^{-2t}$

at t = 0

Current (i) would be minimum

at t = 0

$i_{min} = \frac{10}{3} - \frac{5}{3} - \frac{5}{4} = \frac{5}{12}$

at $t = \infty$, $i \rightarrow$ maximum

$i_{max} = \frac{10}{3}$

Hence $\frac{I_{max}}{I_{min}} = \frac{10}{3} \times \frac{12}{5} = 8.$

Q.18 A metal is heated in a furnace where a sensor is kept above the metal surface to read the power radiated (P) by the metal. The sensor has a scale that displays $\log_2 (P/P_0)$, where P_0 is constant. When the metal surface is at a temperature of 487ºC, the sensor shows a value 1. Assume that the emissivity of the metallic surface remains constant. What is the value displayed by the sensor when the temperature of the metal surface is raised to 2767ºC ?

Ans. **[9]**

Sol. $T_1 = 487 + 273 = 760$ K

$T_2 = 2767 + 273 = 3040$ K

$P = \sigma eAT^4$

$n = \log \frac{\sigma eAT^4}{P_0}$

$1 = \log \frac{\sigma eA(760)^4}{P_0}$ ….(i)

$n = \log \frac{\sigma eA(3040)^4}{P_0}$ ….(ii)

(ii) – (i); $\log\left(\frac{3040}{760}\right)^4 = n - 1$

$\log 4^4 = n - 1$

$8 \log 2 = n - 1$

$8 = n - 1$

$n = 9$

PART II - CHEMISTRY

SECTION – 1 (Maximum Marks : 15)

- This section contains **FIVE** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct.
- For each question, darken the bubble corresponding to the correct option in the ORS.
- For each question, marks will be awarded in one of the following categories. :

Full Marks	: +3	If only the bubble corresponding to the correct option is darkened
Zero Marks	: 0	If none of the bubbles is darkened
Negative Marks	: –1	In all other cases

Q.19 The increasing order of atomic radii of the following Group 13 elements is -

(A) Al < Ga < In < Tl (B) Ga < Al < In < Tl

(C) Al < In < Ga < Tl (D) Al < Ga < Tl < In

Ans. **[B]**

Sol. Order of increasing atomic radii

Ga < Al < In < Tl

Q.20 Among $[Ni(CO)_4]$, $[NiCl_4]^{2-}$, $[Co(NH_3)_4Cl_2]Cl$, $Na_3[CoF_6]$, Na_2O_2 and CsO_2, the total number of paramagnetic compounds is -

(A) 2 (B) 3 (C) 4 (D) 5

Ans. **[B]**

Sol. $[Ni(CO)_4] \longrightarrow$ diamagnetic

$[NiCl_4]^{2-} \longrightarrow$ paramagnetic

$[Co(NH_3)_4Cl_2]Cl \longrightarrow$ diamagnetic

$Na_3[CoF_6] \longrightarrow$ Paramagnetic

$Na_2O_2 \longrightarrow$ diamagnetic

$CsO_2 \longrightarrow$ paramagnetic

Total paramagnetic compounds are 3

Q.21 On complete hydrogenation, natural rubber produces-

(A) ethylene-propylene copolymer

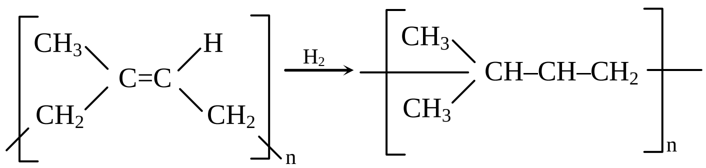
(B) vulcanised rubber

(C) polypropylene

(D) polybutylene

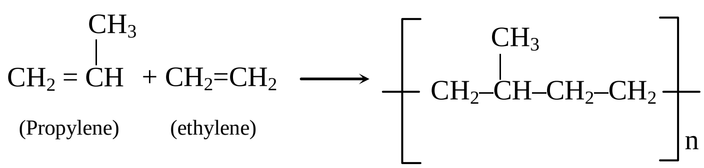
Ans. **[A]**

Sol.

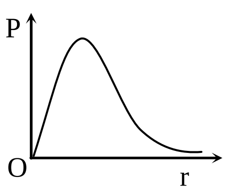


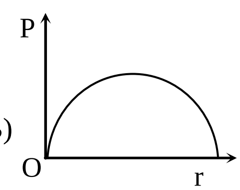
Product

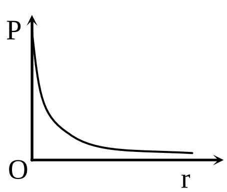
This product can also be obtain by co-polymerisation by ethylene-propylene

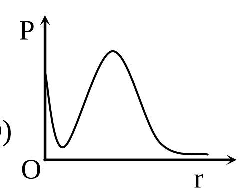


Q.22 P is the probability of finding the 1s electron of hydrogen atom in a spherical shell of infinitesimal thickness, dr, at a distance r from the nucleus. The volume of this shell is $4\pi r^2 dr$, The qualitative sketch of the dependence of P on r is -

(A)


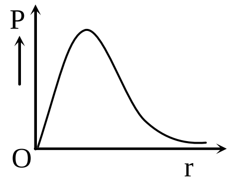
(B)


(C)


(D)


Ans. **[A]**

Sol. For 1s orbital no. of radial nodes is equals to zero, therefore graph of p v/s r is



Q.23 One mole of an ideal gas at 300 K in thermal contact with surroundings expands isothermally from 1.0 L to 2.0 L against a constant pressure of 3.0 atm. In this process, the change in entropy of surroundings (ΔS_{surr}) in JK^{-1} is-

(1 L atm = 101.3 J)

(A) 5.763 (B) 1.013 (C) –1.013 (D) –5.763

Ans. **[C]**

Sol. W.D. = $-P_{ext}(V_2 - V_1)$

W.D. = –3 (2–1) = –3× 101.3 = –303.9J

$\Delta E = q + W$ {$\Delta E = 0$ for isothermal process}

$q = -W$

We know $\Delta S = \frac{q}{T}$

$\Delta S_{system} = +\frac{303.9}{300} = +1.013\ J/K$

$\therefore\ \Delta S_{Surrounding} = -\Delta S_{system}$

$= -1.013$ J/K

SECTION – 2 (Maximum Marks : 32)

Instruction type from Paper

- This section contains **EIGHT** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four options(s) is (are) correct.
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS.
- For each question, marks will be awarded in one of the following categories. :

Full Marks	: +4	If only the bubble(s) corresponding to all the correct option(s) is(are) darkened
Partial Marks	: +1	For darkening a bubble corresponding **to each correct option**. provided NO incorrect option is darkened.
Zero Marks	: 0	If none of the bubbles is darkened
Negative Marks	: –2	In all other cases

- For example, If (A), (C) and (D) are all the correct options for a question, darkening all these three will result in +4 marks; darkening only (A) and (D) will result in +2 marks ; and darkening (A) and (B) will result in –2 marks, as a wrong option is also darkened.

Q.24 The correct statement(s) about the following reaction sequence is(are)-

$$\text{Cumene }(C_9 11_{12}) \xrightarrow[\text{(ii)}H_3O^{\oplus}]{\text{(i)}O_2} \mathbf{P} \xrightarrow{CHCl_3/NaOH} \mathbf{Q}\text{ (major)} + \mathbf{R}\text{ (minor)} \quad \mathbf{Q} \xrightarrow[PhCH_2Br]{NaOH} \mathbf{S}$$

(A) **R** is steam volatile

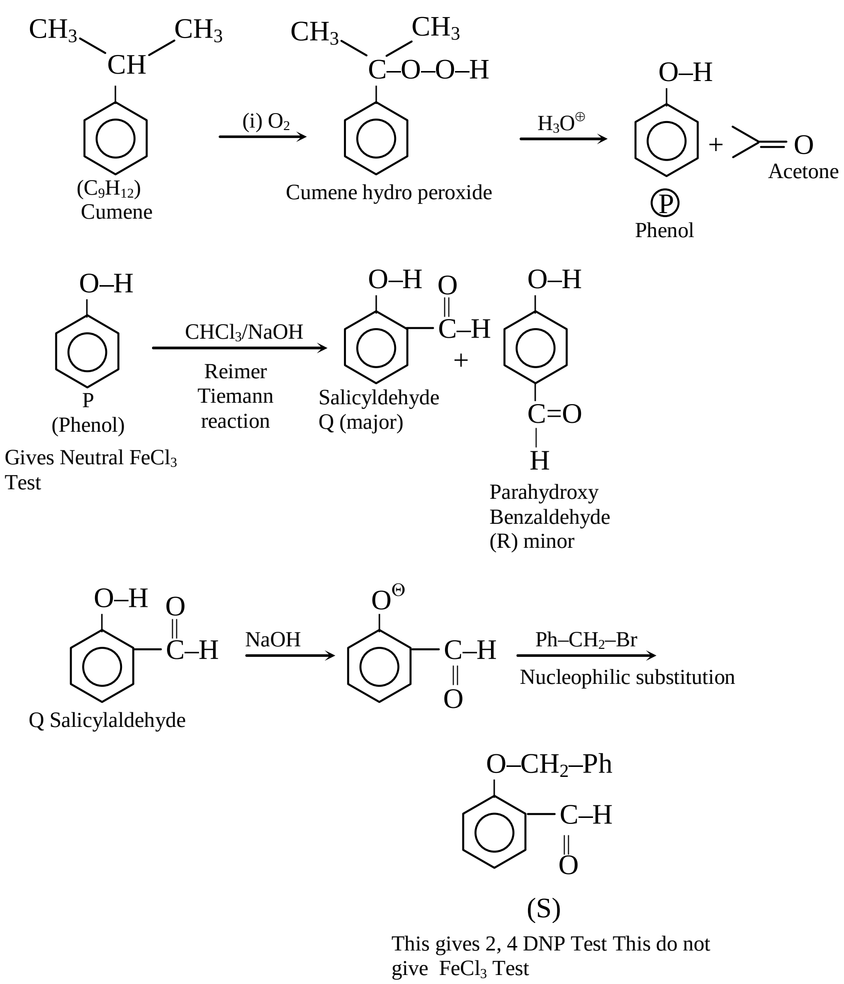
(B) **Q** gives dark violet coloration with 1% aqueous $FeCl_3$ solution

(C) **S** gives yellow precipitate with 2, 4-dinitrophenylhydrazine

(D) S gives dark violet coloration with 1% aqueous $FeCl_3$ solution

Ans. **[B,C]**

Sol.



Q.25 The compound(s) with TWO lone pairs of electrons on the central atom is(are).

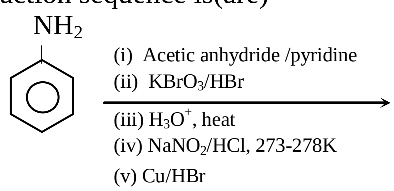
(A) BrF_5 (B) ClF_3 (C) XeF_4 (D) SF_4

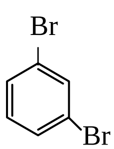
Ans. **[B, C]**

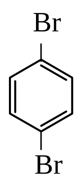
Sol.

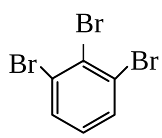
Compounds	Lone pair
BrF_5	1
ClF_3	2
XeF_4	2
SF_4	1

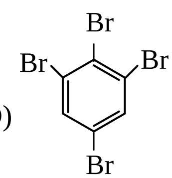
Q.26 The product(s) of following reaction sequence is(are) –



(A) 

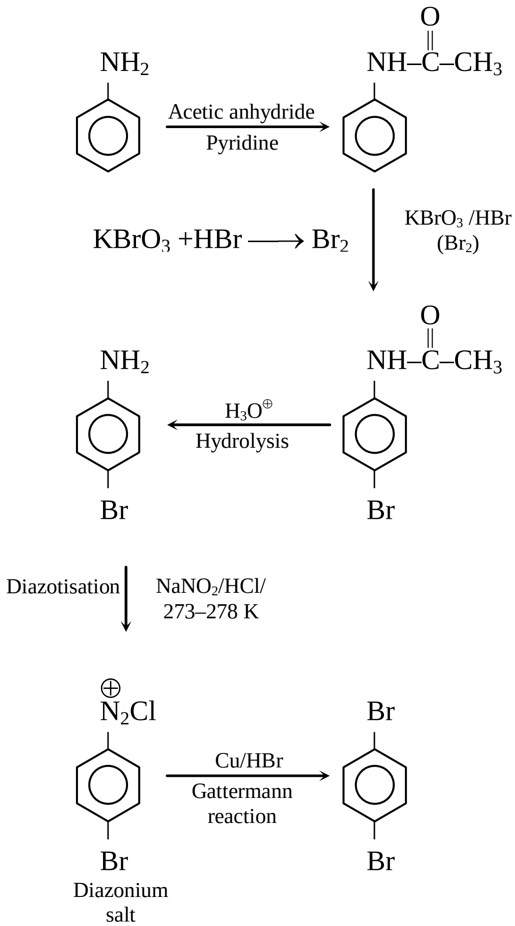
(B) 

(C) 

(D) 

Ans. **[B]**

Sol.



Q.27 According to the Arrhenius equation,

(A) a high activation energy usually implies a fast reaction.

(B) rate constant increases with increase in temperature. This is due to a greater number of collisions whose energy exceeds the activation energy.

(C) higher the magnitude of activation energy, stronger is the temperature dependence of the rate constant

(D) the pre-exponential factor is a measure of the rate at which collisions occur, irrespective of their energy.

Ans. **[B,C,D]**

Sol. (i) Rate of reaction increases with increase in temperature, $K \propto T$

(ii) $\frac{dK}{dT} \propto Ea$

(iii) Pre-exponential factor represents collision frequency

Q.28 The crystalline form of borax has

(A) tetranuclear $[B_4O_5(OH)_4]^{2-}$ unit

(B) all boron atoms in the same plane

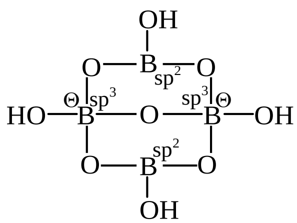
(C) equal number of sp^2 and sp^3 hybridized boron atoms

(D) one terminal hydroxide per boron atom

Ans. **[A,C,D]**

Sol. Crystalline form of borax is $Na_2B_4O_7.10H_2O$ or $Na_2[B_4O_5(OH)_4].8H_2O$

its structure is,

$2Na^+$ 

Q.29 The reagent(s) that can selectively precipitate S^{2-} from a mixture of S^{2-} and SO_4^{2-} in aqueous solution is (are)

(A) $CuCl_2$ (B) $BaCl_2$

(C) $Pb(OOCCH_3)_2$ (D) $Na_2[Fe(CN)_5NO]$

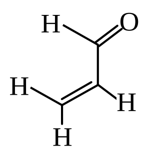
Ans. **[A]**

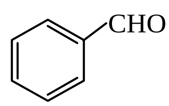
Sol. $CuCl_2 \xrightarrow{S^{-2}} \underset{\text{Black}}{CuS\downarrow}$

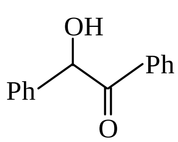
$Na_2[Fe(CN)_5NO] \xrightarrow{S^{-2}} \underset{\text{Violet Colouration}}{Na_4[Fe(CN)_5(NOS)]}$

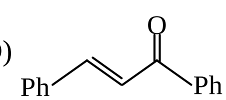
So answer is only (A)

Q.30 Positive Tollen's test is observed for

(A) 

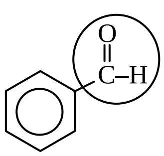
(B) 

(C) 

(D) 

Ans. **[A,B,C]**

Sol. (A) $CH_2=CH-\underset{\|}{\overset{}{C}}-H$ (with =O on C, circled) → Positive Test — Due to presence of aldehyde

(B)  Positive Test — Due to presence of aldehyde

(C) $Ph-\overset{OH}{\overset{|}{CH}}-\underset{\underset{O}{\|}}{C}-Ph$ Positive Test — Because It is a-hydroxy ketone

(D) $Ph-CH=CH-\overset{\overset{O}{||}}{C}-Ph$ Negative Test

Q.31 A plot of the number of neutrons (N) against the number of protons (P) of stable nuclei exhibits upward deviation from linearity for atomic number, Z > 20. For an unstable nucleus having N/P ratio less than 1, the possible mode(s) of decay is (are)

(A) β^- decay (β emission) (B) orbital or K-electron capture

(C) neutron emission (D) β^+ decay (positron emission)

Ans. **[B, D]**

Sol. A nucleus whose low n/p ratio place it below the belt of stability either emits positrons or undergo electron capture

eg. ${}^{11}_{6}C \longrightarrow {}^{11}_{5}B + {}^{0}_{+1}e$ (Positron emission)

eg. ${}^{133}_{56}Ba + {}^{0}_{-1}e \longrightarrow {}^{133}_{55}Cs + X\text{–ray}$

SECTION – 3 (Maximum Marks : 15)

- This section contains **FIVE** questions
- The answer to each question is s **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct answer is darkened.

Zero Marks : 0 In all other cases.

Q.32 The diffusion coefficient of an ideal gas is proportional to its mean free path and mean speed. The absolute temperature of an ideal gas is increased 4 times and its pressure is increased 2 times. As a result, the diffusion coefficient of this gas increases x times. The value of x is.

Ans. **[4]**

Sol. Diffusion constant $\propto \lambda.\nu$

$$\propto \frac{T}{P} \times \sqrt{\frac{8RT}{\pi M}}$$

$$\lambda \propto \frac{T^{3/2}}{P}$$

$$\propto \frac{(4)^{3/2}}{2}$$

$$\propto 4$$

$\therefore$ It will become four times

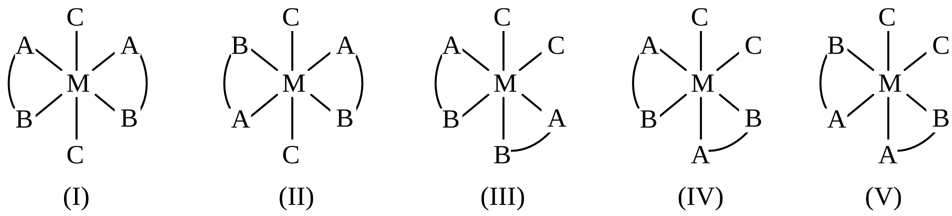
Q.33 The number of geometric isomers possible for the complex $[CoL_2Cl_2]^-$ ($L = H_2NCH_2CH_2O^-$) is

Ans. **[5]**

Sol. $[CoL_2Cl_2]^-$ ($L = H_2NCH_2CH_2O^-$)

General form of complex is $[M(AB)_2C_2]$

Total possible geometrical isomers are 5



Q.34 The mole fraction of a solute in a solution is 0.1. At 298 K, molarity of this solution is the same as its molality . Density of this solution at 298 K is 2.0 g cm^{-3}. The ratio of the molecular weights of the solute and solvent $\left(\frac{MW_{solute}}{MW_{solvent}}\right)$, is

Ans. **[9]**

Sol. $x_B = 0.1$; $m = M$; $d = 2g/cm^3$; $\frac{M_B}{M_A} = ?$

$m = \frac{0.1}{0.9} \times \frac{1000}{M_A}$ [A → Solvent, B → Solute]

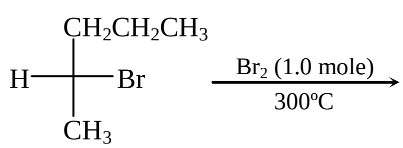
$d = \frac{M}{m} + \frac{MM_B}{1000}$

$2 = 1 + \frac{1}{9} \times \frac{1000}{M_A} . \frac{M_B}{1000}$

$1 = \frac{1}{9} \frac{M_B}{M_A}$

$\frac{M_B}{M_A} = 9$

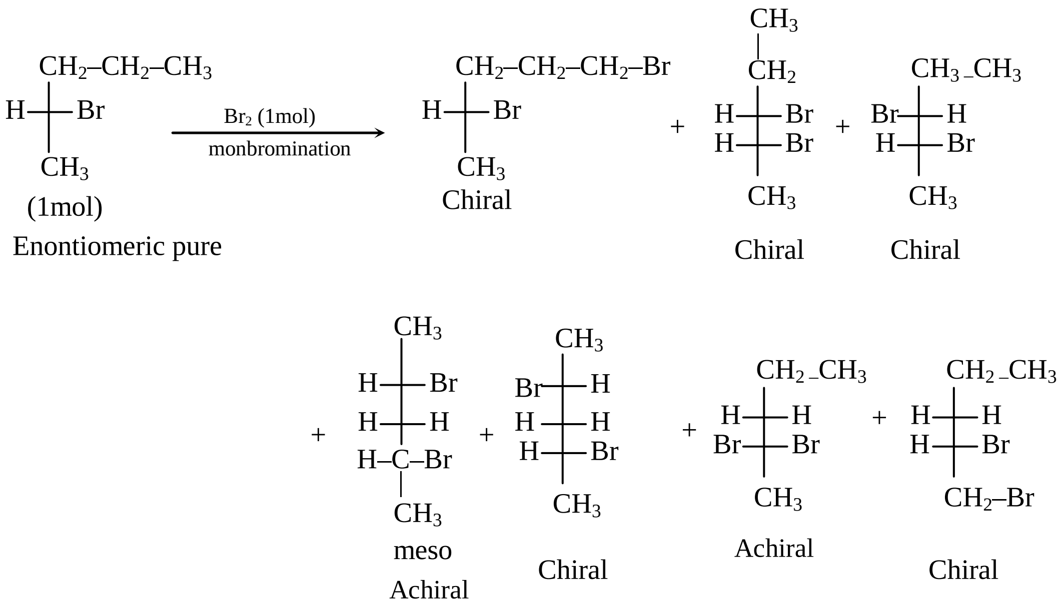
Q.35 In the following monobromination reaction, the number of possible chiral products is



(1.0 mole)
(enantiomerically pure)

Ans. **[5]**

Sol.



If chiral centre is already present, then it's configuration do not change

Q.36 In neutral or faintly alkaline solution, 8 moles of permanganate anion quantitatively oxidize thiosulphate anions to produce X moles of a sulphur containing product. The magnitude of X is

(A) (B) (C) (D)

Ans. **[6]**

Sol.

$$3e^- + MnO_4^- + 2H_2O \longrightarrow MnO_2 + 4OH^-] \times 8$$

$$S_2O_3^{2-} + 10OH^- \longrightarrow 2SO_4^{2-} + 5H_2O + 8e^-] \times 3$$

$$24e^- + 8MnO_4^- + 16H_2O \longrightarrow 8MnO_2 + 32OH^-$$

$$3S_2O_3^{2-} + 3OH^- \longrightarrow 6SO_4^{2-} + 15H_2O + 24e^-$$

$$8MnO_4^- + 3S_2O_3^{2-} + H_2O \longrightarrow 8MnO_2 + 6SO_4^{2-} + 2OH^-$$

8 moles of MnO_4^- ion gives 6 moles of SO_4^{2-} ion

PART III : MATHEMATICS

SECTION – 1 (Maximum Marks : 15)

- This section contains **FIVE** questions.
- Each question has FOUR options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct.
- For each question, darken the bubble corresponding to the correct option in the ORS.
- For each question, marks will be awarded in one of the following categories :

  Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

Zero Marks : 0 If none of the bubbles is darkened.

Negative Marks : –1 In all other cases.

Q.37 A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that

P(computer turns out to be defective given that it is produced in plant T_1)

= 10P (computer turns out to be defective given that it is produced in plant T_2)

where P(E) denotes the probability of an event E. A computer produced in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant T_2 is -

(A) $\frac{36}{73}$ (B) $\frac{47}{79}$ (C) $\frac{78}{93}$ (D) $\frac{75}{83}$

Ans. **[C]**

Sol. Given $P\left(\frac{D}{T_1}\right) = 10P\left(\frac{D}{T_2}\right)$

Now $P(D) = P(T_1)\,P\left(\frac{D}{T_1}\right) + P(T_2)P\left(\frac{D}{T_2}\right)$

$$\frac{7}{100} = \frac{20}{100} \times 10P\left(\frac{D}{T_2}\right) + \frac{80}{100}P\left(\frac{D}{T_2}\right)$$

$$\therefore P\left(\frac{D}{T_2}\right) = \frac{7}{280}, P\left(\frac{D}{T_1}\right) = \frac{70}{280}$$

$$\text{Now } P\left(\frac{T_2}{\overline{D}}\right) = \frac{P(T_2)P\left(\frac{\overline{D}}{T_2}\right)}{P(T_1)P\left(\frac{\overline{D}}{T_1}\right) + P(T_2)P\left(\frac{\overline{D}}{T_2}\right)}$$

$$= \frac{\frac{80}{100}\times\frac{273}{280}}{\frac{20}{100}\times\frac{210}{280}+\frac{80}{100}\times\frac{273}{280}} = \frac{78}{93}$$

Q.38 A debate club consist of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is -

(A) 380 (B) 320 (C) 260 (D) 95

Ans. **[A]**

Sol. Total No. of ways

$= ({}^4C_0 \times {}^6C_4 + {}^4C_1 \times {}^6C_3) \times {}^4C_1$

$= (15 + 80) \times 4$

$= 380$

Q.39 Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

(A) $2(\sec\theta - \tan\theta)$ (B) $2\sec\theta$ (C) $-2\tan\theta$ (D) 0

Ans. **[C]**

Sol. $x^2 - 2x\sec\theta + 1 = 0$

Now roots are $x = \sec\theta \pm \tan\theta$

So $\alpha_1 = \sec\theta - \tan\theta$

& $\beta_1 = \sec\theta + \tan\theta$ $\left\{\begin{array}{l}\text{as } \theta \in \left(-\frac{\pi}{6}, -\frac{\pi}{12}\right) \\ \& \ \alpha_1 > \beta_1\end{array}\right\}$

& $x^2 + 2x\tan\theta - 1 = 0$

Now roots are $x = \pm\sec\theta - \tan\theta$

So, $\alpha_2 = +\sec\theta - \tan\theta$

$\beta_2 = -\sec\theta - \tan\theta$

$\therefore \alpha_1 + \beta_2 = -2\tan\theta$

Q.40 Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\sec x + \text{cosec } x + 2(\tan x - \cot x) = 0$ in the set S is equal to -

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

Ans. **[C]**

Sol. $\sqrt{3}\sec x + \text{cosec } x = 2(\cot x - \tan x)$

$$\frac{\sqrt{3}\sin x + \cos x}{\sin x \cos x} = \frac{2(\cos^2 x - \sin^2 x)}{\sin x \cos x}$$

$\Rightarrow \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \cos 2x$ $(\because \sin x \cos x \neq 0)$

$\Rightarrow \cos\left(x - \frac{\pi}{3}\right) = \cos 2x$

$\Rightarrow 2x = 2n\pi \pm \left(x - \frac{\pi}{3}\right)$, $n \in I$

$\Rightarrow x = 2n\pi - \pi/3$ OR $3x = 2n\pi + \pi/3$

possible solutions in the given interval will be

$x = -\frac{\pi}{3}, -\frac{5\pi}{9}, \frac{\pi}{9}, \frac{7\pi}{9}$

So required sum $= -\frac{\pi}{3} - \frac{5\pi}{9} + \frac{\pi}{9} + \frac{7\pi}{9} = 0$

Q.41 The least value of $\alpha \in R$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$, is -

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

Ans. **[C]**

Sol. $\frac{4\alpha x^2 + \frac{1}{2x} + \frac{1}{2x}}{3} \geq \left(\frac{4\alpha x^2}{4x^2}\right)^{1/3}$ (where $\alpha > 0$ obviously)

$4\alpha x^2 + \frac{1}{x} \geq 3\alpha^{1/3}$

$\therefore 3\alpha^{1/3} \geq 1$

$\alpha \geq \frac{1}{27}$

SECTION – 2 (Maximum Marks : 32)

- This section contains **EIGHT** questions.
- Each question has **FOUR** options (A), (B), (C) and (D) . **ONE OR MORE THAN ONE** of these four option(s) is(are) correct.
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS.
- For each question marks will be awarded in one of the following categories **:**

  Full Marks : +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened.

  Partial Marks : +1 For darkening a bubble corresponding **to each correct option**, provided NO incorrect option is darkened

  Zero Marks : 0 If none of the bubble is darkened.

  Negative Marks : –2 In all other cases.
- For example, if (A), (C) and (D) are all the correct options for a question, darkening all these three will result in +4 marks; darkening only (A) and (D) will result in +2 marks; and darkening (A) and (B) will result in –2 marks, as a wrong option is also darkened.

Q.42 A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$, $x > 0$, passes through the point (1, 3). Then the solution curve

(A) intersects $y = x + 2$ exactly at one point

(B) intersects $y = x + 2$ exactly at two points

(C) intersects $y = (x + 2)^2$

(D) does **NOT** intersect $y = (x + 3)^2$

Ans. **[A,D]**

Sol. $((x + 2)^2 + y(x + 2))\frac{dy}{dx} - y^2 = 0$

$(x + 2)(x + 2 + y)\frac{dy}{dx} = y^2$

$$\frac{dy}{dx} = \frac{y^2}{(x+2)^2 + (x+2)y}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{x+2}{y}\right)^2 + \frac{x+2}{y}}$$

Let $\frac{x+2}{y} = t$

$x + 2 = ty$

$1 = \frac{dy}{dx}.t + y\,\frac{dt}{dx}$

$\left(1 - y\frac{dt}{dx}\right)\frac{1}{t} = \frac{dy}{dx}$

$\therefore\ \left(1 - y\frac{dt}{dx}\right)\frac{1}{t} = \frac{1}{t(t+1)}$

$y\frac{dt}{dx} = \frac{t}{t+1}$

$\frac{x+2}{t}\,\frac{dt}{dx} = \frac{t}{t+1}$

$\int \frac{dx}{x+2} = \int \frac{t+1}{t^2}dt$

$\ell n(x + 2) = \ell n\, t - \frac{1}{t} + C$

$\ell n\left(\frac{x+2}{t}\right) = -\frac{1}{t} + C$

$\Rightarrow \ell n\, y = \frac{-y}{x+2} + C \qquad \left(\because y = \frac{x+2}{t}\right)$

It passes through (1, 3)

$\ell n\, 3 = -1 + C \Rightarrow C = 1 + \ell n\, 3$

$\ell n\, y = \frac{-y}{x+2} + 1 - \ell n\, 3$

$\ell n\, \frac{y}{3} = \frac{-y}{x+2} + 1$

This curve intersects $y = x + 2$

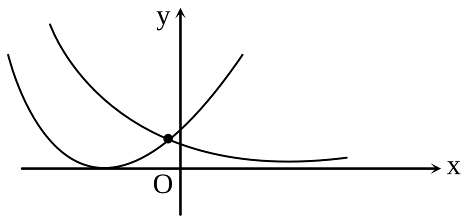
$\ell n\, \frac{x+2}{3} = 0 \qquad \Rightarrow x = -1$

Only one solution

This curves intersects $y = (x + 2)^2$

$\ell n \dfrac{(x+2)^2}{3} = -x-2+1 = -x-1$

$(x+2)^2 = 3e^{-x-1}$ but $x > 0$ No solution



$y = (x+3)^2$

$\ell n \dfrac{(x+3)^2}{3} = \dfrac{-(x+3)^2}{x+2} + 1 = \dfrac{-x^2-9-6x+x+2}{x+2}$

$\ell n \dfrac{(x+3)^2}{3} = \dfrac{-(x^2+5x+7)}{x+2}$

↓ positive ↓ negative as $x > 0$

Q.43 Consider a pyramid OPQRS located in the first octant ($x \ge 0, y \ge 0, z \ge 0$) with O as origin, and OP and OR along the x-axis and the y-axis respectively. The base OPQR of the pyramid is a square with OP = 3. The point S is directly above the mid-point T of diagonal OQ such that TS = 3. Then

(A) the acute angle between OQ and OS is $\dfrac{\pi}{3}$

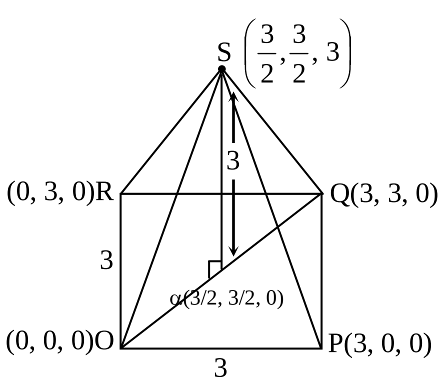
(B) the equation of the plane containing the triangle OQS is $x - y = 0$

(C) the length of the perpendicular from P to the plane containing the triangle OQS is $\dfrac{3}{\sqrt{2}}$

(D) the perpendicular distance from O to the straight line containing RS is $\sqrt{\dfrac{15}{2}}$

Ans. **[B,C,D]**

Sol.



acute angle between OQ & OS

(A) Distance between OT = $\frac{3}{\sqrt{2}}$

Distance between ST = 3

Clearly angle between OT and OS

$\tan\theta = \frac{3}{3/\sqrt{2}} = \sqrt{2}$

$\tan\theta = \sqrt{2}$

$\theta = \tan^{-1}(\sqrt{2})$

Option A is wrong

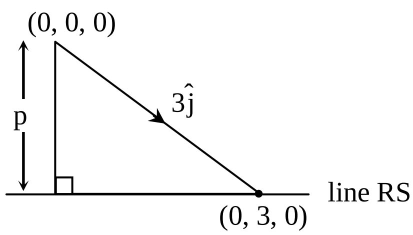
(B) Clearly equation of plane OQS is x – y = 0

(for each point x = y)

(C) Equation of plane OQS is x – y = 0

Distance from P(3, 0, 0) = $\frac{|3-0|}{\sqrt{1+1}} = \frac{3}{\sqrt{2}}$

(D)



$$\overrightarrow{\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}}$$

p = projection of (3j) in $\perp$ direction $\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}$

$$p = \frac{\left|3\hat{j}\times\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)\right|}{\left|\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right|} = \frac{\left|-\frac{9}{2}\hat{k}+9\hat{i}\right|}{\left|\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right|}$$

$$\Rightarrow \frac{\sqrt{\frac{81}{4}+81}}{\sqrt{\frac{9}{4}+\frac{9}{4}+9}} = \frac{\sqrt{\frac{9}{4}+9}}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}} = \frac{\sqrt{45}}{\sqrt{6}} = \frac{\sqrt{15}}{\sqrt{2}}$$

Q.44 The circle $C_1 : x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centre Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then

(A) $Q_2Q_3 = 12$

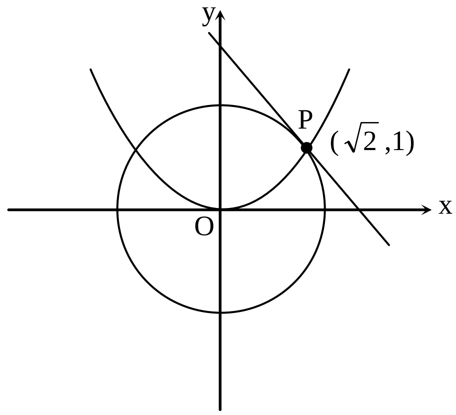
(B) $R_2R_3 = 4\sqrt{6}$

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$

(D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

Ans. **[A,B,C]**

Sol. Solving the equation of circle and parabola we get point P($\sqrt{2}$,1)



Now equation of tangent at P($\sqrt{2}$, 1) is $\sqrt{2}$ x + y = 3

Let the centre of the circle C_2 and C_3 are Q_2 (0, k_1) and Q_3(0, k_2)

Tangent also touches these circles, so

$$\left|\frac{k_1 - \sqrt{3}}{\sqrt{2+1}}\right| = 2\sqrt{3}$$

$\therefore$ k_1 = 9, –3

So, Q_2(0, 9) and Q_3(0, –3)

$\therefore$ $Q_2Q_3 = 12$

Let the point of contact R_2 is (α_1, β_1)

tangent at (α_1, β_1) for circle C_2

$\alpha_1 x + \beta_1 y - 9(y + \beta_1) = -69$

Comparing it with $\sqrt{2}$ x + y = 3

$$\frac{\alpha_1}{\sqrt{2}} = \frac{\beta_1 - 9}{1} = \frac{9\beta_1 - 69}{3}$$

$\alpha_1 = -2\sqrt{2}$, $\beta_1 = 7$

$\therefore$ R_2 ($-2\sqrt{2}$, 7)

Similarly let R_3 is (α_2, β_2)

Equation of tangent at (α_2, β_2)

$\alpha_2 x + \beta_2 y + 3(y + \beta_2) = 3$

Comparing with $\sqrt{2}$ x + y = 3

$$\frac{\alpha_2}{\sqrt{2}} = \frac{\beta_2 + 3}{1} = \frac{3 - 3\beta_2}{3}$$

$\Rightarrow$ $\alpha_2 = 2\sqrt{2}$, $\beta_2 = -1$

$\therefore$ R_3 ($2\sqrt{2}$, –1)

$R_2R_3 = \sqrt{(4\sqrt{2})^2 + 64} = \sqrt{96} = 4\sqrt{6}$

Area of the triangle OR_2R_3

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2\sqrt{2} & 7 \\ 0 & 2\sqrt{2} & -1 \end{vmatrix} = 6\sqrt{2}$$

Area of the triangle PQ_2Q_3

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 1 & 9 & -3 \end{vmatrix} = 6\sqrt{2}$$

Q.45 Let $f : R \to R$, $g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$ (C) $h(0) = 16$ (D) $h(g(3)) = 36$

Ans. **[B,C]**

Sol. $f(x) = x^3 + 3x + 2$

$g(f(x)) = x$

$g'(f(x))\ f'(x) = 1$

$g'(2) = \frac{1}{f'(0)} = \frac{1}{(3x^2+3)_{x=0}} = \frac{1}{3}$

$h(g(g(x))) = x$

replace x by f(x)

$h(g(g(f(x)))) = f(x)$

$h(g(x)) = f(x)$

replace x by f(x)

$h(g(f(x))) = f(f(x))$

$h(x) = f(f(x))$

$h(x) = (x^3 + 3x + 2)^3 + 3(x^3 + 3x + 2) + 2$

$h'(x) = 3(x^3 + 3x + 2)^2 + (3x^2 + 3) + 9x^2 + 9 + 0$

$h'(1) = 3(6)^2\,(6) + 9 + 9$

$= 18 \times 36 + 18$

$= 648 + 18 = 666$

$\because h(x) = f(f(x))$

$h(0) = f(f(0))$

$h(0) = f(2)$

$= 8 + 6 + 2$

$h(0) = 16$

$\because h(g(x)) = f(x)$

$h(g(3)) = f(3)$

$= 27 + 9 + 2 = 38$

Q.46 Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in R$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then

(A) $\alpha = 0, k = 8$

(B) $4\alpha - k + 8 = 0$

(C) $\det(P\ \text{adj}(Q)) = 2^9$

(D) $\det(Q\ \text{adj}(P)) = 2^{13}$

Ans. **[B,C]**

Sol. $PQ = kI$

$Q = kP^{-1} = \frac{k\,\text{adj}\,P}{|\overline{P}|}$

$= \frac{k}{12\alpha + 20} = \begin{bmatrix} 5\alpha & 10 & -\alpha \\ 3\alpha & 6 & -(3\alpha+4) \\ -10 & 12 & \alpha \end{bmatrix}$

$q_{23} = -\frac{k}{8} = \frac{k}{12\alpha + 20}\ (-3\alpha - 4)$

$12\alpha + 20 = 24\alpha + 32$

$-12 = 12\alpha$

$\alpha = -1$

$Q = \frac{k}{8}\begin{bmatrix} -5 & 10 & 1 \\ -3 & 6 & -1 \\ -10 & 12 & 2 \end{bmatrix}$

$|Q| = \frac{k^2}{2} = \frac{k^3}{8^3}\ (-5(12 + 12) - 10(-6 - 10) + 1\ (-36 + 60))$

$\frac{1}{2} = \frac{k}{8.8.8}(-120 + 160 + 24)$

$\frac{1}{2} = \frac{k}{8}$

$k = 4$

$|P\ \text{adj}\ Q|$

$= |P\ \text{adj}(4P^{-1})|$

$= |16\ P\ \text{adj}\ P^{-1}|$

$= \left|16P.\frac{P}{|P|}\right| = \frac{16^3}{|P|^3}\ |P^2|$

$= \frac{16^3}{|P|} = \frac{16^3}{8} = 2^9$

$|Q.\text{adj}\ P|$

$= |Q\ \text{adj}(kQ^{-1}|$

$= \left| k^2 Q . \frac{Q}{|Q|} \right|$

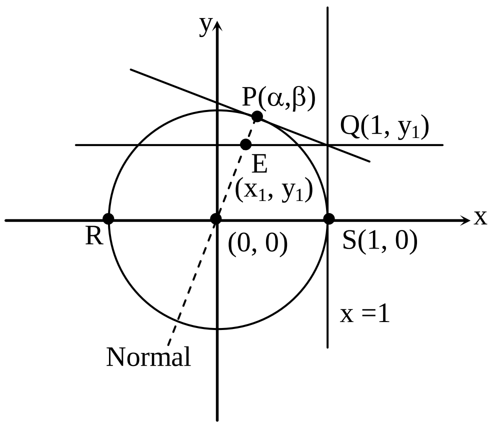
$= 2^3 |Q|^2 = 2^3 . 2^6 = 2^9$

Q.47 Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1, 0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. Then the locus of E passes through the point(s)

(A) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (C) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Ans. **[A,C]**

Sol.



Point P is (α, β)

Let Point E is (x_1, y_1)

given circle is $x^2 + y^2 = 1$ … (1)

$\therefore \alpha^2 + \beta^2 = 1$

Also tangent at $P(\alpha, \beta)$ is

$\alpha x + \beta y = 1$

Put (1, y)

$\alpha + \beta y_1 = 1$ … (2)

Now Slope of OP = Slope of OE

$\frac{\beta}{\alpha} = \frac{y_1}{x_1}$

$\alpha = \frac{x_1 \beta}{y_1}$ … (3)

Put in (2)

$\beta = \frac{y_1}{x_1 + y_1^2}$

$\therefore \alpha = \frac{x_1}{x_1 + y_1^2}$

Put α, β in (1) we get

$$\frac{x_1^2}{(x_1+y_1^2)^2} + \frac{y_1^2}{(x_1+y_1^2)^2} = 1$$

$\Rightarrow y_1^2 = 1 - 2x_1$

Q.48 Let $f : (0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then

(A) $\lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = 1$

(B) $\lim_{x\to 0^+} xf\left(\frac{1}{x}\right) = 2$

(C) $\lim_{x\to 0^+} x^2 f'(x) = 0$

(D) $|f(x)| \leq 2$ for all $x \in (0, 2)$

Ans. **[A]**

Sol. $f : (0, \infty) \longrightarrow R$

$f'(x) = 2 - \frac{f(x)}{x} \qquad \forall\, x \in (0, \infty)$

$\frac{dy}{dx} = 2 - \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = 2$

I.F. $= e^{\int \frac{1}{x} dx} = e^{\ell nx} = x$

$\Rightarrow xy = \int 2x\,dx$

$\Rightarrow xy = x^2 + c$

$\Rightarrow y = x + \frac{c}{x}$

$\therefore f(x) = x + \frac{c}{x}$

$f(1) = 1 + c \neq 1 \qquad \therefore c \neq 0$

$f'(x) = 1 - \frac{c}{x^2}$

$f'\left(\frac{1}{x}\right) = 1 - cx^2$

$\lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = \lim_{x\to 0^+} (1 - cx^2) = 1$

$\lim_{x\to 0^+} x f\left(\frac{1}{x}\right) = \lim_{x\to 0^+} x\left(\frac{1}{x} + cx\right)$

$= \lim_{x\to 0^+} (1 + cx^2) = 1$

$\lim_{x\to 0^+} x^2 f'(x) = \lim_{x\to 0^+} (x^2 - c) = -c \neq 0$

$f(x) = x + \frac{c}{x}$

$\lim_{x\to 0^+} f(x) \to \pm\infty$

So f(x) is not bounded

So $|f(x)| \le 2$ not possible

Q.49 In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z, respectively, and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then

(A) area of the triangle XYZ is $6\sqrt{6}$

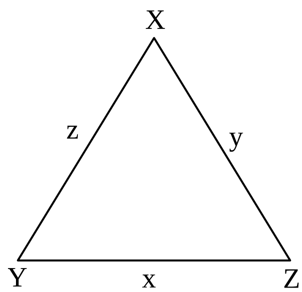
(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

Ans. **[A,C,D]**

Sol.



$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = k$ (Let)

$\Rightarrow x = s - 4k,\ y = s - 3k,\ z = s - 2k$

Now $2s = x + y + z = 3s - 9k$

$\therefore k = \frac{s}{9}$

Area of triangle

$\Delta = \sqrt{s(s-x)(s-y)(s-z)}$

$= \sqrt{s \cdot 4k \cdot 3k \cdot 2k} = \sqrt{24\frac{s^4}{729}}$

$= \frac{2}{9}\sqrt{\frac{2}{3}}s^2$

Area of incircle $\pi r^2 = \frac{8\pi}{3}$

$\therefore r = 2\sqrt{\frac{2}{3}} = \frac{\Delta}{s} = \frac{2\sqrt{2}}{9\sqrt{3}}s$

$\therefore s = 9$ and $k = 1$

$\therefore$ Area of triangle $\Delta = \frac{2\sqrt{2}}{9\sqrt{3}} \times 81 = 6\sqrt{6}$

$$R = \frac{xyz}{4\Delta} = \frac{5\times6\times7}{4\times6\sqrt{6}} = \frac{35}{24}\sqrt{6}$$

$$\sin\frac{X}{2}\ \sin\frac{Y}{2}\ \sin\frac{Z}{2} = \sqrt{\frac{(s-y)(s-z)}{yz}\frac{(s-x)(s-z)}{xz}\frac{(s-x)(s-y)}{xy}}$$

$$= \frac{(s-x)(s-y)(s-z)}{xyz}$$

$$= \frac{4\times3\times2}{5\times6\times7} = \frac{4}{35}$$

$$\sin^2\left(\frac{X+Y}{2}\right) = \frac{1-\cos(X+Y)}{2}$$

$$= \frac{1-\cos(\pi-Z)}{2} = \frac{1+\cos Z}{2}$$

$$= \frac{1}{2}+\frac{1}{2}\left(\frac{x^2+y^2-z^2}{2xy}\right)$$

$$= \frac{1}{2}+\frac{1}{2}\left(\frac{25+36-49}{2\times5\times6}\right) = \frac{6}{10} = \frac{3}{5}$$

SECTION – 3 (Maximum Marks : 15)

- This section contains **FIVE** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- For each question, marks will be awarded in one of the following categories :

  Full Marks : +3 If only the bubble corresponding to the correct answer is darkened.

  Zero Marks : 0 In all other cases.

Q.50 Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is

Ans. **[5]**

Sol. $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$

$$= \frac{(1+x)^2[(1+x)^{48}-1]}{(1+x)-1} + (1+mx)^{50}$$

$$= \frac{(1+x)^{50}}{x} - \frac{(1+x)^2}{x} + (1+mx)^{50}$$

given that coefficient of $x^2 = (3n + 1)\ {}^{51}C_3$

$\therefore\ {}^{50}C_3 + {}^{50}C_2\ m^2 = (3n + 1)\ {}^{51}C_3$

after simplifying

$$\Rightarrow \frac{{}^{50}C_3}{{}^{50}C_2} + m^2 = (3n+1)\frac{{}^{51}C_3}{{}^{50}C_2}$$

$\Rightarrow \frac{50-2}{3} + m^2 = (3n+1)\frac{51}{3}$

$\Rightarrow 16 + m^2 = 17\,(3n + 1)$

$\Rightarrow m^2 = 51n + 1$

m is the smallest +ve integer

$\therefore$ n = 5

Q.51 Let $\alpha, \beta \in R$ be such that $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x - \sin x} = 1$. Then $6(\alpha + \beta)$ equals

Ans. **[7]**

Sol. $\lim_{x\to 0}\frac{x^2\sin\beta x}{\alpha x - \sin x} = 1$

$\Rightarrow \lim_{x\to 0}\frac{x^2\left(\frac{\sin\beta x}{\beta x}\right)\beta x}{\alpha x - \left(x - \frac{x^3}{\lfloor 3} + ...\right)} = 1$

$\Rightarrow \lim_{x\to 0}\frac{\beta x^3}{(\alpha-1)x + \frac{x^3}{6} + ...} = 1$

So $\alpha - 1 = 0,$ $6\beta = 1$

$\therefore \alpha = 1,$ $\beta = \frac{1}{6}$

Hence $6(\alpha + \beta) = 6\left(1 + \frac{1}{6}\right) = 7$

Q.52 Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and r, s $\in$ {1, 2, 3}. Let $P = \begin{bmatrix}(-z)^r & z^{2s}\\ z^{2s} & z^r\end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is

Ans. **[1]**

Sol. $z = \omega$

$P^2 = \begin{bmatrix}(-\omega)^r & \omega^{2s}\\ \omega^{2s} & \omega^r\end{bmatrix}\begin{bmatrix}(-\omega)^r & \omega^{2s}\\ \omega^{2s} & \omega^r\end{bmatrix}$

$P^2 = \begin{bmatrix}\omega^{2r} + \omega^{4s} & (-\omega)^r\omega^{2s} + \omega^{2s}\omega^r\\ \omega^{2s}(-\omega)^r + \omega^r\omega^{2s} & \omega^{4s} + \omega^{2r}\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

$\therefore (-1)^r\,\omega^r\,\omega^{2s} + \omega^{2s}\,\omega^r = 0$

So r = 1, 3

Also $\omega^{2r} + \omega^{4s} = -1$

when r = 1 $\omega^2 + \omega^{4s} = -1$

$\therefore$ s = 1

when r = 3 $\omega^6 + \omega^{4s} = -1$

$1 + \omega^{4s} = -1$

Not possible.

Q.53 The total number of distinct $x \in \mathbb{R}$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

Ans. **[2]**

Sol. $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$

$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ 2x & 4x^2 & 1 \\ 3x & 9x^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 2x & 4x^2 & 8x^3 \\ 3x & 9x^2 & 27x^3 \end{vmatrix} = 10$

$\Rightarrow x^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 9 & 1 \end{vmatrix} + x^6 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 10$

$\Rightarrow 2x^3 + 12x^6 = 10$

$\Rightarrow 6x^6 + x^3 - 5 = 0$

$\Rightarrow (6x^3 - 5)(x^3 + 1) = 0$

$\Rightarrow x^3 = \frac{5}{6}, \qquad x^3 = -1$

So two real roots

Q.54 The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$ is

Ans. **[1]**

Sol. Let $g(x) = \int_0^x \frac{t^2}{1+t^4} dt - (2x) + 1$

Let $I(x) = \int_0^x \frac{t^2}{1+t^4} dt$

$= \frac{1}{2} \int_0^x \frac{2t^2}{1+t^4} dt$

$$= \frac{1}{2}\int_0^x \left(\frac{t^2+1}{1+t^4} + \frac{t^2-1}{1+t^4}\right)dt$$

$$= \frac{1}{2}\int_0^x \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt + \frac{1}{2}\int_0^x \frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)^2-2}$$

$$I(x) = \frac{1}{2}\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)\right]_0^x + \frac{1}{2}\times\frac{1}{2\sqrt{2}}\left[\ell n\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|\right]_0^x$$

Now, $g'(x) = \frac{x^2}{1+x^4} - 2$

$g'(x) = \frac{1}{\frac{1}{x^2}+x^2} - 2$ since $\frac{1}{x^2}+x^2 \ge 2$

$\therefore$ $g'(x) < 0$ So $\frac{1}{x^2+\frac{1}{x^2}} \le \frac{1}{2}$

Now, $g(0) = 1$

& $g(1) = \int_0^1 \frac{t^2}{1+t^4}dt - 2 + 1$

$$= \int_0^1 \frac{t^2}{1+t^4}dt - 1$$

$$= \left[0 + \frac{1}{4\sqrt{2}}\ell n\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\right] - \left[\frac{1}{2\sqrt{2}}\left(-\frac{\pi}{2}\right) + \frac{1}{4\sqrt{2}}\ell n\left(\frac{1}{1}\right)\right] - 1$$

$$= \frac{1}{4\sqrt{2}}\ell n\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) + \frac{\pi}{4\sqrt{2}} - 1$$

g(1) = – ve

$\therefore$ $g(0) > 0$ = & $g(1) < 0$

& $g'(x) < 0$

$\therefore$ g(x) has one root in [0, 1]