The solution of the given differential equation is :

Put y = vx

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = -\frac{vx}{x}\frac{\left(\frac{x^2}{3(vx)^2} + 1\right)}{\left(\frac{(vx)^2}{3x^2} + 1\right)} = -v\frac{\left(\frac{1}{3(v)^2} + 1\right)}{\left(\frac{(v)^2}{3} + 1\right)} = -\frac{1 + 3(v)^2}{3 + (v)^2} \times \frac{1}{v}$$
$$= -\frac{1 + 3(v)^2}{3v + (v)^3}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1 + 3(v)^2}{3v + (v)^3} - v = -\frac{1 + 3(v)^2 + 3(v)^2 + (v)^4}{3v + (v)^3}$$
$$= \frac{1 + 6(v)^2 + (v)^4}{3v + (v)^3}$$

$$\Rightarrow \frac{3v + (v)^3}{1 + 6(v)^2 + (v)^4}dv = -\frac{dx}{x}$$

Integrating both the sides we get:

$$\Rightarrow \int \frac{3v + (v)^3}{1 + 6(v)^2 + (v)^4}dv = -\int \frac{dx}{x} + c$$

$$\Rightarrow \frac{\ln|1 + 6(v)^2 + (v)^4|}{4} + \ln|x| = \ln|c|$$

Resubstituting the value of y = vx we get

$$\Rightarrow \frac{\ln\left|1 + 6(\frac{y}{x})^2 + (\frac{y}{x})^4\right|}{4} + \ln|x| = \ln|c|$$

$\Rightarrow y^4 + 6x^2y^2 + x^4 = C$

Ans: $y^4 + 6x^2y^2 + x^4 = C$

19. Question

In each of the following differential equation show that it is homogeneous and solve it.

$(x - \sqrt{xy})\,dy = ydx$

Answer

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x - \sqrt{xy}} = \frac{1}{\frac{x}{y} - \sqrt{\frac{x}{y}}} = \frac{1}{\left(\frac{y}{x}\right)^{-1} - \sqrt{\left(\frac{y}{x}\right)^{-1}}}$$

$$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{1}{\left(\frac{vx}{x}\right)^{-1} - \sqrt{\left(\frac{vx}{x}\right)^{-1}}} = \frac{1}{\frac{1}{v} - \frac{1}{\sqrt{v}}} = \frac{v\sqrt{v}}{\sqrt{v} - v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\sqrt{v}}{\sqrt{v}-v} - v = \frac{v\sqrt{v} - v\sqrt{v} + v^2}{\sqrt{v}-v} = \frac{v^2}{\sqrt{v}-v}$$

$$\Rightarrow \frac{\sqrt{v}-v}{v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{v^{\frac{3}{2}}}dv - \frac{1}{v}dv = \frac{dx}{x}$$

Integrating both the sides we get:

$$\Rightarrow \int \frac{1}{v^{\frac{3}{2}}}dv - \int \frac{1}{v}dv = \int \frac{dx}{x} + c$$

$$\Rightarrow \frac{-1}{\sqrt{v}} - \ln|v| = \ln|x| + c$$

Resubstituting the value of y = vx we get

$$\Rightarrow \frac{-1}{\sqrt{\left(\frac{y}{x}\right)}} - \ln\left(\frac{y}{x}\right) = \ln|x| + c$$

$$\Rightarrow 2\sqrt{\frac{x}{y}} + \log|y| = C$$

Ans: $2\sqrt{\frac{x}{y}} + \log|y| = C$

20. Question

In each of the following differential equation show that it is homogeneous and solve it.

$$x^2\frac{dy}{dx} + y^2 = xy$$

Answer

$$\Rightarrow \frac{dy}{dx} = \frac{xy - y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

$$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \left(\frac{vx}{x}\right)^2 = v - v^2$$

$$\Rightarrow x\frac{dv}{dx} = -v^2$$

$$\Rightarrow \frac{dv}{-v^2} = \frac{dx}{x}$$

Integrating both the sides we get:

$$\Rightarrow \int \frac{dv}{-v^2} = \int \frac{dx}{x} + c$$

$$\Rightarrow \frac{1}{v} = \ln|x| + c$$

Resubstituting the value of y = vx we get

$$\Rightarrow \frac{1}{\frac{y}{x}} = \ln|x| + c$$

$$\Rightarrow \frac{x}{y} = \ln|x| + \ln|c|$$

$$\Rightarrow \frac{x}{y} = \ln|xc|$$

$$\Rightarrow e^{\frac{x}{y}} = xc$$

Ans: $e^{\frac{x}{y}} = xc$

21. Question

In each of the following differential equation show that it is homogeneous and solve it.

$x\frac{dy}{dx} = y$ (log y - log x + 1)

Answer

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\log\left(\frac{y}{x}\right) + 1\right)$$

$$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{vx}{x}\left(\log\left(\frac{vx}{x}\right) + 1\right) = v(\log(v) + 1)$$

$$\Rightarrow x\frac{dv}{dx} = v\log v$$

$$\Rightarrow \frac{dv}{v\log v} = \frac{dx}{x}$$

Integrating both the sides we get:

$$\Rightarrow \int \frac{dv}{v\log v} = \int \frac{dx}{x} + c$$

$$\Rightarrow \log|\log v| = \log|xc|$$

$$\Rightarrow \log|v| = xc$$

$$\Rightarrow v = e^{xc}$$

Resubstituting the value of y = vx we get

$\Rightarrow y = xe^{xc}$

Ans: $y = xe^{xc}$

22. Question

In each of the following differential equation show that it is homogeneous and solve it.

$x\frac{dy}{dx} - y + x\sin\frac{y}{x} = 0$

Answer

$\Rightarrow x\frac{dy}{dx} - y + x\sin\frac{y}{x} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{y - x\sin\frac{y}{x}}{x} = \frac{y}{x} - \sin\frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{vx}{x} - \sin\frac{vx}{x} = v - \sin v$

$\Rightarrow x\frac{dv}{dx} = -\sin v$

$\Rightarrow \frac{dv}{\sin v} = -\frac{dx}{x}$

Integrating both the sides we get:

$\Rightarrow \int \frac{dv}{\sin v} = -\int \frac{dx}{x} + c$

$\Rightarrow \log\tan(\frac{v}{2}) = -\log|x| + c$

Resubstituting the value of y = vx we get

$\Rightarrow \log\tan\left(\frac{y}{2x}\right) = -\log|x| + \log c$

$\Rightarrow X\tan\left(\frac{y}{2x}\right) = C$

Ans: $X\tan\left(\frac{y}{2x}\right) = C$

23. Question

In each of the following differential equation show that it is homogeneous and solve it.

$x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$

Answer

$$\Rightarrow \frac{dy}{dx} = \frac{y - x\cos^2\left(\frac{y}{x}\right)}{x} = \left(\frac{y}{x}\right) - \cos^2\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \left(\frac{vx}{x}\right) - \cos^2\left(\frac{vx}{x}\right) = v - \cos^2 v$$

$$\Rightarrow x\frac{dv}{dx} = -\cos^2 v$$

$$\Rightarrow \frac{dv}{\cos^2 v} = -\frac{dx}{x}$$

Integrating both the sides we get:

$$\Rightarrow \int \frac{dv}{\cos^2 v} = -\int \frac{dx}{x} + c$$

$$\Rightarrow \tan v = -\ln|x| + c$$

Resubstituting the value of y = vx we get

$$\Rightarrow \tan\left(\frac{y}{x}\right) + \ln|x| = c$$

Ans: $\tan\left(\frac{y}{x}\right) + \ln|x| = c$

24. Question

In each of the following differential equation show that it is homogeneous and solve it.

$$\left(x\cos\frac{y}{x}\right)\frac{dy}{dx} = \left(y\cos\frac{y}{x}\right) + x$$

Answer

$$\Rightarrow \left(x\cos\frac{y}{x}\right)\frac{dy}{dx} = y\cos\frac{y}{x} + x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\cos\frac{y}{x} + x}{x\cos\frac{y}{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{vx}{x} + \sec\frac{vx}{x} = v + \sec v$$

$$\Rightarrow x\frac{dv}{dx} = \sec v$$

$$\Rightarrow \frac{dv}{\sec v} = \frac{dx}{x}$$

Integrating both the sides we get:

$$\Rightarrow \int \frac{dv}{\sec v} = \int \frac{dx}{x} + c$$

⇒ sinv = ln|x| + c

Resubstituting the value of y = vx we get

$$\Rightarrow \sin\left(\frac{y}{x}\right) = \ln|x| + c$$

Ans: $\sin\left(\frac{y}{x}\right) = \ln|x| + c$

25. Question

Find the particular solution of the different equation.$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$, it being given that y = 2 when x = 1

Answer

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2} = \frac{y}{x} + \frac{y^2}{2x^2}$$

$$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{vx}{x} + \frac{(vx)^2}{2x^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2}{2}$$

$$\Rightarrow \frac{dv}{v^2} = \frac{2dx}{x}$$

Integrating both the sides we get:

$$\Rightarrow \int \frac{dv}{v^2} = 2\int \frac{dx}{x} + c$$

$$\Rightarrow \frac{-1}{v} = 2\ln|x| + c$$

Resubstituting the value of y = vx we get

$\Rightarrow \frac{-x}{y} = 2\ln|x| + c$

Now,

y = 2 when x = 1

$\Rightarrow \frac{-1}{2} = 2\ln|1| + c$

$\Rightarrow c = \left(-\frac{1}{2}\right) \Rightarrow y = \frac{2x}{(1-\log|x|)}$

Ans: $y = \frac{2x}{(1-\log|x|)}$

26. Question

Find the particular solution of the differential equation $\left\{x\sin^2\frac{y}{x} - y\right\}$dx + xdy = 0, it being given that y = $\frac{\pi}{4}$ when x = 1.

Answer

$\Rightarrow \frac{dy}{dx} = \frac{y - x\sin^2\left(\frac{y}{x}\right)}{x} = \left(\frac{y}{x}\right) - \sin^2\left(\frac{y}{x}\right)$

$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \left(\frac{y}{x}\right) - \sin^2\left(\frac{y}{x}\right) = v - \sin^2 v$

$\Rightarrow x\frac{dv}{dx} = -\sin^2 v$

$\Rightarrow \frac{dv}{\sin^2 v} = -\frac{dx}{x}$

Integrating both the sides we get:

$\Rightarrow \int \frac{dv}{\sin^2 v} = -\int \frac{dx}{x} + c$

$\Rightarrow \cot v = \ln|x| + c$

Resubstituting the value of y = vx we get

$\Rightarrow \cot\left(\frac{y}{x}\right) = \ln|x| + c$

y = $\frac{\pi}{4}$ when x = 1

$$\Rightarrow \cot\left(\frac{\frac{\pi}{4}}{1}\right) = \ln|1| + c$$

⇒ c = 1

Ans: $\cot\left(\frac{y}{x}\right) = \ln|x| + 1$

27. Question

Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{y(2y - x)}{x(2y + x)}$ given that y = 1 when x = 1.

Answer

$$\Rightarrow \frac{dy}{dx} = \frac{y(2y - x)}{x(2y + x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\left(2\frac{y}{x} - 1\right)}{x\left(2\frac{y}{x} + 1\right)}$$

$$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{vx\left(2\frac{vx}{x} - 1\right)}{x\left(2\frac{vx}{x} + 1\right)} = v\left(\frac{2v - 1}{2v + 1}\right)$$

$$\Rightarrow x\frac{dv}{dx} = v\left(\frac{2v - 1}{2v + 1}\right) - v$$

$$\Rightarrow x\frac{dv}{dx} = v\left(\frac{2v - 1 - 2v - 1}{2v + 1}\right) \Rightarrow x\frac{dv}{dx} = \frac{-2v}{2v + 1}$$

$$\Rightarrow \frac{2v + 1}{2v}dv = \frac{-dx}{x}$$

$$\Rightarrow dv + \left(\frac{1}{2v}\right)dv = \frac{-dx}{x}$$

Integrating both the sides we get:

$$\Rightarrow \int\left(dv + \left(\frac{1}{2v}\right)dv\right) = -\int\frac{dx}{x} + c$$

$$\Rightarrow v + \frac{\ln|v|}{2} = -\ln|x| + c$$

Resubstituting the value of y = vx we get

$$\Rightarrow \frac{y}{x} + \frac{\ln\left|\frac{y}{x}\right|}{2} = -\ln|x| + c$$

y = 1 when x = 1

1 + 0 = - 0 + c

$\Rightarrow$ c = 1

$\Rightarrow \frac{y}{x} + \frac{1}{2}\log|xy|$ = 1

Ans: $\frac{y}{x} + \frac{1}{2}\log|xy|$ = 1

28. Question

Find the particular solution of the differential equation $xe^{y/x}$ - y + $x\frac{dy}{dx}$ = 0, given that y(1) = 0.

Answer

$$\Rightarrow\ xe^{\frac{y}{x}} - y\ +\ x\frac{dy}{dx}\ =\ 0$$

$$\Rightarrow\ x\frac{dy}{dx}\ =\ y - xe^{\frac{y}{x}}$$

$$\Rightarrow\ \frac{dy}{dx}\ =\ \left(\frac{y}{x}\right) - e^{\frac{y}{x}}$$

$$\Rightarrow\ \frac{dy}{dx}\ =\ f\left(\frac{y}{x}\right)$$

$\Rightarrow$ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$$\Rightarrow\ \frac{dy}{dx}\ =\ v\ +\ x\frac{dv}{dx}$$

$$v\ +\ x\frac{dv}{dx}\ =\ \left(\frac{vx}{x}\right) - e^{\frac{vx}{x}}$$

$$\Rightarrow\ x\frac{dv}{dx}\ =\ -e^{v}$$

$$\Rightarrow\ \frac{dv}{e^{v}}\ =\ \frac{-dx}{x}$$

Integrating both the sides we get:

$$\Rightarrow\ \int\frac{dv}{e^{v}}\ =\ -\int\frac{dx}{x}\ +\ c$$

$$\Rightarrow\ -e^{-v}\ =\ -\ln|x|\ +\ c$$

Resubstituting the value of y = vx we get

$$\Rightarrow\ -e^{-\left(\frac{y}{x}\right)}\ =\ -\ln|x|\ +\ c$$

Now,y(1) = 0

$$\Rightarrow\ -e^{-(0)}\ =\ -\ln|1|\ +\ c$$

$$\Rightarrow\ c\ =\ -1$$

$\Rightarrow\ \log|x| + e^{-y/x}$ = 1

Ans: $\log|x| + e^{-y/x}$ = 1

29. Question

Find the particular solution of the differential equation $xe^{y/x} - y + x\frac{dy}{dx} = 0$, given that y(e) = 0.

Answer

$\Rightarrow xe^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0$

$\Rightarrow x\frac{dy}{dx} = y - xe^{\frac{y}{x}}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) - e^{\frac{y}{x}}$

$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \left(\frac{vx}{x}\right) - e^{\frac{vx}{x}}$

$\Rightarrow x\frac{dv}{dx} = -e^{v}$

$\Rightarrow \frac{dv}{e^{v}} = \frac{-dx}{x}$

Integrating both the sides we get:

$\Rightarrow \int \frac{dv}{e^{v}} = -\int \frac{dx}{x} + c$

$\Rightarrow -e^{-v} = -\ln|x| + c$

Resubstituting the value of y = vx we get

$\Rightarrow -e^{-\left(\frac{y}{x}\right)} = -\ln|x| + c$

Now,y(e) = 0

$\Rightarrow -e^{-(0)} = -\ln|e| + c$

$\Rightarrow c = 0$

⇒ y = - xlog(log|x|)

Ans: y = - xlog(log|x|)

30. Question

The slope of the tangent to a curve at any point (x,y) on it is given by $\frac{y}{x} - \left(\cot\frac{y}{x}\right)\left(\cos\frac{y}{x}\right)$, where x>0 and y>0. If the curve passes through the point $\left(1, \frac{\pi}{4}\right)$, find the equation of the curve.

Answer

It is given that:

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \cot\frac{y}{x}\cos\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

⇒ the given differential equation is a homogenous equation.

The solution of the given differential equation is :

Put y = vx

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \cot\frac{vx}{x}\cos\frac{vx}{x}$$

$$\Rightarrow x\frac{dv}{dx} = -\cot v\cos v$$

$$\Rightarrow \frac{dv}{-\cot v\cos v} = \frac{dx}{x}$$

Integrating both the sides we get:

$$\Rightarrow \int \frac{dv}{-\cot v\cos v} = \int \frac{dx}{x} + c$$

$$\Rightarrow \frac{-1}{\cos v} = \ln|x| + c$$

Resubstituting the value of y = vx we get

$$\Rightarrow \frac{-1}{\cos\frac{y}{x}} = \ln|x| + c$$

the curve passes through the point $\left(1, \frac{\pi}{4}\right)$

$$\Rightarrow \frac{-1}{\cos\frac{\frac{\pi}{x}}{1}} = \ln|1| + c$$

$$\Rightarrow c = -\sqrt{2}$$

$$\Rightarrow \sec\frac{y}{x} + \log|x| = \sqrt{2}$$

Ans:The equation of the curve is: $\sec\frac{y}{x} + \log|x| = \sqrt{2}$